Xalq ta’limi vazirligi Fan olimpiadalari bo‘yicha iqtidorli o‘quvchilar bilan ishlash departamentining matematika fanidan 9 – haftalik topshiriqlarining yechimlari

10 – 11 sinf o‘quvchilari uchun

1 – masala. Tenglamalar sistemasining barcha musbat yechimlarini toping:

$$\begin{cases} a\sqrt{b} - c = a \\ b\sqrt{c} - a = b \\ c\sqrt{a} - b = c \end{cases}$$

Yechimi: Dastlab a, b, c sonlarning barchasi birdan katta ekanligini ko‘rsatamiz. Teskarisini faraz qilaylik: ulardan birortasi, masalan $c \le 1$ bo‘lsin. U holda

$$b = b\sqrt{c} - a \le b - a < b$$

bo‘lib, ziddiyatga kelamiz. Ko‘rishimiz mumkin a, b, c sonlaridan qaysidir ikkitasining qiymati 4 bo‘lsa, u holda uchinchisining ham qiymati 4 ga tengligi kelib chiqadi. Aytaylik a, b, c sonlaridan bittasi, masalan $a = 4$ bo‘lsin. U holda

$$c = c\sqrt{a} - b = 2c - b$$

ya’ni $b = c$ va 1-tenglamadan

$$16b = \left(a\sqrt{b}\right)^2 = (a+b)^2 = (b+4)^2$$

yoki

$$a = b = c = 4$$

bo‘ladi. Demak biz

$$(a-4)(b-4)(c-4) \ne 0$$

bo‘lgan holni qarashimiz yetarli. Drixle prinsipiga ko‘ra a, b, c sonlaridan qaysidir ikkitasi bir vaqtda yoki 4 dan katta, yoki 4 dan kichik. Masalan ular a va b bo‘lsin. Quyidagi hollarni qaraylik.

1-hol: a, $b>4$ bo‘lsin. U holda $a=a\sqrt{b}-c>2a-c$, ya’ni $c>a>4$. Xuddi shu usulda $a>b>c>a$ isbotlashimiz mumkin. Ziddiyat!

2-hol: a, $b<4$ bo‘lsin. U holda $a=a\sqrt{b}-c<2a-c$, ya’ni $c<a<4$. Xuddi shu usulda $a<b<c<a$ isbotlashimiz mumkin. Ziddiyat! Demak masalaning yagona yechimi $a=b=c=4$ ekan.

2 – masala. O‘tkir burchakli $\triangle ABC$ uchburchakning BM va CN balandliklari H nuqtada kesishadi. Uning BC tomonida W nuqta olingan. $\triangle BWN$ uchburchakning tashqi chizilgan aylanasi w_1 va WX unda diametr. Xuddi shunday $\triangle CWM$ uchburchakning tashqi chizilgan aylanasi w_2 va WY unda diametr. U holda X, Y, H nuqtalar bir to‘g‘ri chiziqda yotishini isbotlang.

Yechimi: Ravshanki,

$$\angle ANC=\angle HNB=90^\circ$$

va

$$\angle A=90^\circ-\angle ABM=\angle NHB.$$

Demak $\triangle NHB\sim\triangle NAC$ ekan. Shu bilan birga,

$$\angle NXB=180^\circ-\angle NWB=\angle NWC$$

va WX diametr bo‘lgani uchun

$$\angle XNB=90^\circ-\angle BNW=\angle WNC$$

demak $\triangle XNB\sim\triangle WNC$ ekan. Bu uchburchaklarning o‘xshashligidan,

$$\angle NAC=\angle NHB,\ \angle NXB=\angle NWC,$$

$$\angle ANW=\angle ANC+\angle CNW=\angle HNB+\angle BNX=\angle XNH$$

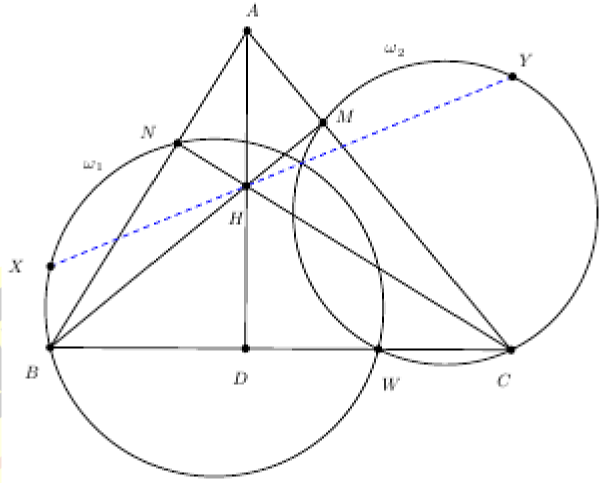
va

$$\frac{NA}{AC}=\frac{NH}{HB}$$

tenglik kelib chiqadi. Shu sababli *TTBBB* ga ko‘ra $XNHB$ va $WNAC$ to‘rtburchaklar o‘xshash va

$$\angle NHX = \angle NAW = \angle WAB$$

bo‘ladi.



Demak $YMHC$ va $WMAB$ to‘rtburchaklarning o‘xshash ekanligini va bundan

$$\angle WAB = \angle YHC$$

natijani olamiz. U holda

$$\angle NHX = \angle YHC$$

ekan. Bu tenglikdan X, Y, H nuqtalar bir to‘g‘ri chiziqda yotishi kelib chiqadi.

3 – masala. Aytaylik a va b natural sonlar uchun $\frac{(4a^2-1)^2}{4ab-1}$ ifoda butun son bo‘lsin. U holda $a=b$ tenglikni isbotlang.

Yechimi: $4ab-1 \mid (4a^2-1)^2$ ga ko‘ra

$$4ab-1 \mid b^2(4a^2-1)^2 - (4ab-1)(4a^3b-2ab+a^2) = (a-b)^2$$

munosabatni topamiz. Teskarisini faraz qilaylik.

$$4ab-1 \mid (a-b)^2$$

shartni qanoatlantiradigan turli natural a va b natural sonlar mavjud bo‘lsin

va

$$k = \frac{(a-b)^2}{4ab-1} > 0$$

deb olaylik. Fiksirlangan $k \in \mathbb{N}$ uchun

$$S=\left\{(a,b):(a,b)\in\mathbb{N}\times\mathbb{N}\,\middle|\,k=\frac{(a-b)^2}{4ab-1}\right\}$$

to‘plamni qaraylik. Shunday $(A,B)\in S$ juftlikni qaraylik, bunda $A+B$ yig‘indi eng kichik qiymatga erishsin. Umumiylikka zarar yetkazmasdan $A>B$ bo‘lsin. Quyidagi kvadrat tenglamani qaraylik:

$$k=\frac{(x-B)^2}{4xB-1} \text{ yoki } x^2-(2B+4kb)x+B^2+k=0.$$

Bu tenglama $x_1=A$ va x_2 ildizlarga ega. Viyet formulasiga ko‘ra

$$x_2=2B+4kB-A=\frac{B^2+k}{A}$$

tenglikni topamiz. Demak x_2 musbat butun son. U holda $(x_2,B)\in S$ va $A+B$ yig‘indini minimalligiga ko‘ra

$$x_2\geq A \text{ yoki } \frac{B^2+k}{A}\geq A.$$

U holda

$$\frac{(A-B)^2}{4AB-1}=k\geq A^2-B^2=(A-B)(A+B)$$

yoki

$$A+B>A-B\geq(4AB-1)(A+B)>A+B$$

ziddiyat. Demak $a=b$.

4 – masala. Barcha $n>1$ natural sonlarni topingki, quyidagi tenglama turli toq natural sonlarda yechimga ega bo‘lsin.

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_{n-1}}+\frac{1}{x_n}=1.$$

Yechimi: Umumiylikka zarar yetkazmasdan $1<x_1<x_2<...<x_n$ deb olaylik. Tenglamani quyidagi ko'rinishda yozib olamiz:

$$x_1x_2...x_n = x_2..x_n + ... + x_1x_2...x_{n-1}$$

Ko'rishimiz mumkin chap tomon toq son va o'ng tomon n ta toq son yig'indisidan iborat. Demak n ham toq son. Quyidagi

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}<1$$

tengsizlikka ko'ra $1<n\le 6$ da yechim yo'q. Aytaylik $n=7$ bo'lsin. x_1 dan x_5 gacha sonlarni eng katta qiymatini olsak:

$$\frac{2}{x_6}>\frac{1}{x_6}+\frac{1}{x_7}=1-\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\right)>\frac{2}{17}$$

Demak $11<x_6<17$. U holda $x_6\in\{13,15\}$. Lekin ikkala holni ham tekshirsak yechim bo'lmaydi, ya'ni bu holda ham yechim yo'q. Biz induksiya orqali $n\ge 9$ toq natural son uchun yechim mavjudligini ko'rsatamiz. Qo'shimchasiga $3|x_n$ bo'ladi. Masalan $n=9$ uchun

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{21}+\frac{1}{165}+\frac{1}{693}=1$$

Aytaylik $x_1,\ x_2,\dots,\ x_n$ toq sonlar **(*)** tenglamaning yechimi bo'lsin. U holda

$$y_1=x_1,\ y_2=x_2,\ y_3=x_3,\ \dots\ y_{n-1}=x_{n-1},\ y_n=\frac{5}{3}x_n,\ y_n=3x_n \text{ va } y_{n+2}=15x_n \vdots 3$$

kabi olsak n dan keyingi toq son $n+2$ uchun yechimlar mavjudligi keladi.

5 – masala. Bir nechta natural sonlar bitta qatorga yozilgan. Davron ikkita ketma – ket yozilgan x va y natural sonlarni qaraydi. Agar x son y ning chap tomonida yozilgan va $x>y$ bo'lsa, u holda (x, y) ning o'rniga $(y+1, x)$ yoki $(x-1, x)$ ni yozadi. U holda Davron bu operatsiyani chekli marta bajara olishini isbotlang.

Isboti: Ko'rish mumkin operatsiyadan keyin ham qatordagi sonlarning eng kattasi o'zgarmay qoladi. Aytaylik qatorda dastlab $a_1,\ a_2, ..., a_n$ sonlar yozilgan bo'lsin. Ular uchun

$$S = a_1 + 2a_2 + \ldots + na_n \quad \textbf{(*)}$$

yig'indini qaraylik. Ketma-ket yozilgan (a_i, a_{i+1}) sonlarni qaraylik, bu yerda $a_i > a_{i+1}$. U (c, a_i) juftlikka almashtiriladi, bu yerda $c = a_{i+1} + 1$ yoki $c = a_i - 1$. U holda avvalgi va keyingi **(*)** yig'indilar orasidagi farq

$$d = (ic + (i+1)a_i) - (ia_i + (i+1)a_{i+1}) = a_i - a_{i+1} + i(c - a_{i+1}) \geq 1$$

ga teng bo'ladi. Boshqa tomondan

$$S \leq (1 + 2 + \ldots + n)M$$

tengsizlik o'rinli, bu yerda $M = \max a_i$. Demak har bir operatsiyada S yig'indi oshadi, lekin u yuqoridan chegaralangan. Demak Davron chekli operatsiyalarni bajarishi mumkin.

7 – 9 sinf o'quvchilari uchun

1 – masala. Aytaylik musbat x, y, z sonlar uchun quyidagi shart o'rinli bo'lsin:

$$\max\{|x-y|, |x-y|, |x-y|\} < 2$$

U holda quyidagi tengsizlikni isbotlang:

$$\sqrt{xy+1} + \sqrt{yz+1} + \sqrt{zx+1} > x + y + z.$$

Yechimi: Berilgan $|x-y| < 2$ shartga ko'ra

$$x^2 - 2xy + y^2 < 4$$

yoki

$$x^2 + 2xy + y^2 < 4 + 4xy = 4(1+xy)$$

tengsizlikni topamiz. Demak

$$x + y < 2\sqrt{1+xy}$$

tengsizlik o'rinli ekan. Xuddi shu usulda

$$y + z < 2\sqrt{1+yz}$$

va

$$z + x < 2\sqrt{1+zx}$$

munosbatlarni topishimiz mumkin. Bu tengsizliklarni qo'shib natijani olamiz.

2 – masala. Aytaylik $a=\frac{1}{2}\sqrt{\sqrt{2}+\frac{1}{8}}-\frac{\sqrt{2}}{8}$ bo‘lsin. U holda $a^2+\sqrt{a^4+a+1}$ ifodaning qiymatini toping.

Yechimi: Masala shartdan

$$\left(a+\frac{\sqrt{2}}{8}\right)^2=\frac{1}{4}\left(\sqrt{2}+\frac{1}{8}\right) \text{ yoki } a^2=\frac{\sqrt{2}}{4}(1-a)$$

tenglikni topib, bundan esa

$$a^4=\left(\frac{\sqrt{2}}{4}(1-a)\right)^2=\frac{a^2-2a+1}{8}$$

tenglikni topamiz. U holda

$$a^4+a+1=\frac{a^2-2a+1}{8}+a+1=\frac{a^2+6a+9}{8}=\frac{(a+3)^2}{8}$$

yoki

$$a^2+\sqrt{a^4+a+1}=\frac{\sqrt{2}}{4}(1-a)+\frac{a+3}{\sqrt{8}}=\sqrt{2}$$

ekan.

Javob: $\sqrt{2}$

3 – masala. Musbat a, b, c sonlar uchun $\aleph=\left\{a; b; c; \frac{a^2}{b}; \frac{b^2}{c}; \frac{c^2}{a}\right\}$ to‘plamni qaraylik. Bu to‘plamda aynan uchta turli son bo‘lishi mumkinmi?

Yechimi: Aytaylik a, b, c sonlari orasida turlilari topilsin. Aniqlik uchun

$$a=\max\{a,b,c\}$$

deb olaylik. Agar $a>b$ yoki $a=b>c$ bo‘lsa, u holda mos ravishda

$$\frac{a^2}{b}>a \text{ va } \frac{b^2}{c}>b=a$$

bo‘ladi. Ya’ni

$$\left\{\frac{a^2}{b};\frac{b^2}{c};\frac{c^2}{a}\right\}$$

sonlarining eng kattasi, $\{a,b,c\}$ sonlarining eng kattasidan katta. Xuddi shu usulda

$$\left\{\frac{a^2}{b};\frac{b^2}{c};\frac{c^2}{a}\right\}$$

sonlarining eng kichigi, $\{a,b,c\}$ sonlarining eng kichigidan kichikligini isbotlash mumkin. Demak kamida 4 ta turli son mavjud. Agar a, b, c sonlari barchasi teng bo‘lsa, u holda

$$a=b=c=\frac{a^2}{b}=\frac{b^2}{c}=\frac{c^2}{a}$$

ya’ni bittagina turli sondan iborat. Ya’ni hech qachon aynan 3 ta turli son mavjud emas.

4 – masala. $\frac{4!}{0!}+\frac{5!}{1!}+\frac{6!}{2!}+...+\frac{2020!}{2016!}+\frac{2021!}{2017!}$ yig‘indini hisoblang.

Yechimi: Quyidagi ayniyatni qaraylik:

$$\frac{(k+4)!}{k!}=(k+1)(k+2)(k+3)(k+4)=k^4+10k^3+35k^2+50k+24$$

Bundan tashqari quyidagi ma’lum yig‘indilarni qaraylik:

$$S_1(n)=0+1+2+...+n=\frac{n(n+1)}{2}$$

$$S_2(n)=0^2+1^2+2^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_3(n)=0^3+1^3+2^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

$$S_4(n)=0^4+1^4+2^4+...+n^4=\frac{6n^5+15n^4+10n^3-n}{30}$$

U holda masalada so‘ralgan yig‘indi quyidagiga teng:

$$S_4(2017)+10S_3(2017)+35S_2(2017)+50S_1(2017)+24$$

(mustaqil soddalashtiring!)

5 – masala. Berilgan $\triangle ABC$ uchburchakda $\angle C=90^\circ$. Aytaylik AC tomonda D nuqta, BD kesmada esa K nuqta olingan, bunda $\angle ABC=\angle KAD=\angle AKD$ tenglik o‘rinli bo‘ladi. U holda $BK=2DC$ tenglikni isbotlang.

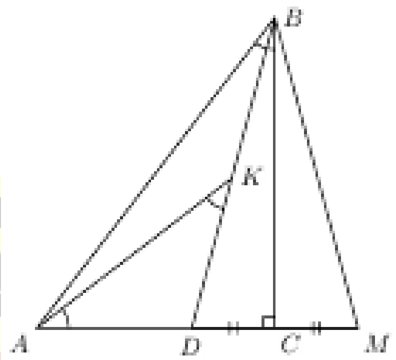
Yechimi: DC kesmaning davomida $CM=CD$ bo‘ladigan M nuqta olaylik. U holda teng yonli uchburchak xossasidan $BD=BM$ bo‘ladi. Biz berilgan shartlardan foydalanib,

$$\angle BAK=\angle AKD-\angle ABK=\angle ABC-\angle ABK=\angle KBC=\angle CBM$$

tenglikni va bundan esa

$$\angle BAM = \angle BAK + \angle KAC = \angle CBM + \angle ABC = \angle ABM$$

munosabatni topamiz.



Demak $\triangle ABM$ teng yonli uchburchak, ya'ni $AM = BM$. U holda

$$BK = BD - KD = BM - KD = AM - KD = AM - AD = DM = 2DC$$

Fan olimpiadalari bo'yicha iqtidorli o'quvchilar bilan ishlash departamenti sizga omad tilaydi!