Xalq ta'limi vazirligi Fan olimpiadalari bo'yicha iqtidorli o'quvchilar bilan ishlash departamentining matematika fanidan haftalik topshiriqlarning yechimlari

10-11 sinf o'quvchilari uchun

1. Xasan doskaga $2\sqrt{2}, \sqrt{2}, -2\sqrt{2}$ sonlarini yozdi. Xusan esa har qadamda ulardan ixtiyoriy ikkitasi a va b larni tanlab, ularning o'rniga $\frac{a+b}{\sqrt{2}}$ va $\frac{a-b}{\sqrt{2}}$ sonlarini yozadi. Xusan nechtadir qadamdan keyin doskada $2, 3, 4$ sonlarini hosil qilishi mumkinmi?

Yechimi: Quyidagi ayniyatni qaraylik:

$$\left(\frac{a+b}{\sqrt{2}}\right)^2 + \left(\frac{a-b}{\sqrt{2}}\right)^2 = a^2 + b^2 \quad \textbf{(1)}$$

Demak ixtiyoriy qadamda doskadagi sonlarning kvadratlari yig'indisi o'zgarmas ekan. Teskarisini faraz qilaylik. Ya'ni

$$\left\{2\sqrt{2}, \sqrt{2}, -2\sqrt{2}\right\} \Rightarrow \{2, 3, 4\} \quad \textbf{(2)}$$

mumkin bo'lsin. U holda

$$S_0 = (2\sqrt{2})^2 + (\sqrt{2})^2 + (2\sqrt{2})^2 = 18 \quad \textbf{(3)}$$

va

$$S_k = 2^2 + 3^2 + 4^2 = 29 \quad \textbf{(4)}$$

teng bo'lishi zarur, ammo bu ziddiyat. Demak farazimiz noto'g'ri.

Javob: Hosil qilish mumkin emas.

2. Perimetri 2020 ga teng va tomonlari natural sonlardan iborat nechta uchburchak mavjud?

Yechimi: Biz quyidagi teoremadan foydalanamiz:

Teorema: Tomonlari uzunliklari natural son bo'lib, perimetri berilgan n soniga teng bo'lgan uchburchaklar soni:

$$T(n)=\begin{cases}\tau\left(\dfrac{n^2}{48}\right), agar, n-juft\\ \tau\left(\dfrac{(n+3)^2}{48}\right), agar, n-toq\end{cases}$$

Bu yerda $\tau(x)-x$ ga eng yaqin butun son. Masalan, $\tau(2,1)=2, \tau(3,6)=4$.

Teoremaning isbotini https://t.me/bazarbaevs/19 havolada berilgan materialdan o‘rganishingiz mumkin.

Demak so‘ralgan uchburchaklar soni

$$T(2020)=\tau\left(\frac{2020^2}{48}\right)=\tau(85008,333...)=85008 \quad \textbf{(5)}$$

Javob: 85008 ta.

3. O‘zaro teng bo‘lmagan a, b, c haqiqiy sonlar uchun quyidagi tengsizlikni isbotlang:

$$\left(\frac{a-b}{b-c}-2\right)^2+\left(\frac{b-c}{c-a}-2\right)^2+\left(\frac{c-a}{a-b}-2\right)^2\geq 17$$

Yechimi: Quyidagi belgilashlarni olaylik:

$$\frac{a-b}{b-c}=x, \frac{b-c}{c-a}=y, \frac{c-a}{a-b}=z \quad \textbf{(6)}$$

U holda

$$xyz=1 \quad \textbf{(7)}$$

tenglik o‘rinli. Bundan tashqari

$$x+1=\frac{a-b}{b-c}+1=\frac{a-c}{b-c} \quad \textbf{(8)}$$

Tenglikka ko‘ra

$$(x+1)(y+1)(z+1)=\frac{a-c}{b-c}\cdot\frac{b-a}{c-a}\cdot\frac{c-b}{a-b}=-1 \quad \textbf{(9)}$$

Demak

$$-1=(x+1)(y+1)(z+1)=xyz+x+y+z+xy+yz+zx+1\Rightarrow$$

$$x+y+z+xy+yz+zx=-3 \quad \textbf{(10)}$$

Masala shartida talab qilingan tengsizlik

$$(x-2)^2+(y-2)^2+(z-2)^2\geq 17\Rightarrow$$

$$x^2+y^2+z^2-4(x+y+z)-5\geq 0 \quad \textbf{(11)}$$

tengsizlikka teng kuchli. **(10)** tenglikka ko‘ra

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)=(x+y+z)^2-$$

$$2(-3-x-y-z)=(x+y+z)^2+2(x+y+z)+6 \quad \textbf{(12)}$$

U holda **(11)** tengsizlik quyidagi tengsizlikka teng kuchli

$$x^2+y^2+z^2-4(x+y+z)-5\geq 0 \Rightarrow (x+y+z)^2-2(x+y+z)+1\geq 0 \quad \textbf{(13)}$$

Bu esa $(x+y+z-1)^2\geq 0$ tengsizlikka teng kuchli. Tengsizlik isbotlandi.

Izoh: Tengsizlikda tenglik holi x, y, z sonlar $t^3-t^2-4t-1=0$ kub tenglamaning ildizlari bo‘lganda bajariladi.

4. Tenglamalar sistemasining barcha haqiqiy yechimlarini toping:

$$\begin{cases} a^2+b^2+c^2=a^3+b^3+c^3 \\ a^3b+b^3c+c^3a=3 \end{cases}$$

Yechimi: Quyidagi lemmadan foydalanamiz:

Lemma: Barcha haqiqiy a, b, c sonlar uchun quyidagi tengsizlik o‘rinli:

$$(a^2+b^2+c^2)^2\geq 3(a^3b+b^3c+c^3a) \quad \textbf{(14)}$$

Isboti: Qavslarni ochib, ixchamlashtirishdan so‘ng

$$(a^2-ab+2bc-b^2-ac)^2+(b^2-bc+2ca-c^2-ab)^2+(c^2-ca+2ab-a^2-bc)^2\geq 0 \quad \textbf{(15)}$$

tengsizlikka kelamiz. Lemma isbotlandi.

Lemma va masala shartiga ko‘ra

$$a^2+b^2+c^2\geq 3 \quad \textbf{(16)}$$

tengsizlikni hosil qilamiz.

Bundan tashqari

$$(a-1)^2(2a+1)^2\geq 0\Rightarrow 2a^3+1\geq 3a^2 \quad \textbf{(17)}$$

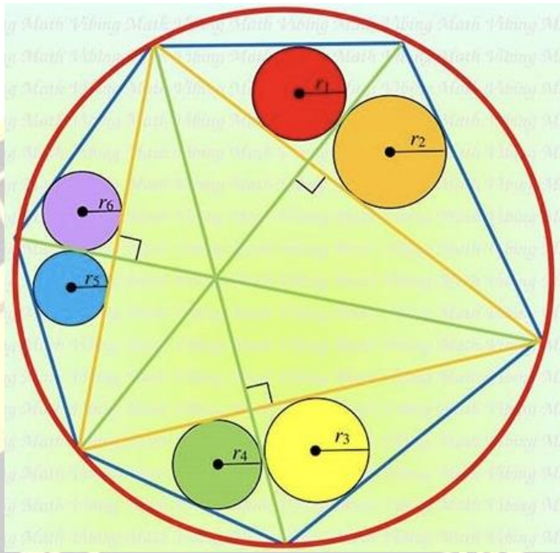
tengsizlikka ko‘ra

$$2(a^3+b^3+c^3)+3\geq 3(a^2+b^2+c^2)\Rightarrow\ 3\geq a^2+b^2+c^2 \quad \textbf{(18)}$$

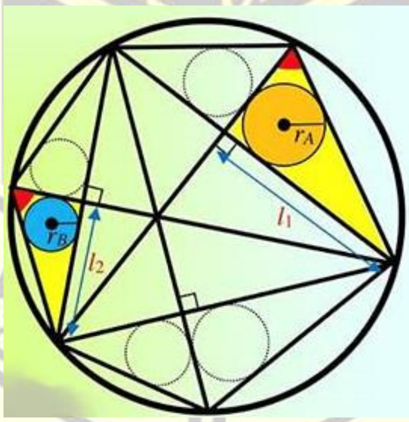
Demak $a=b=c=1$ ekan.

Javob: $a=b=c=1$

5. Quyidagi chizmaga asosan $r_1r_3r_5=r_2r_4r_6$ tenglikni isbotlang.



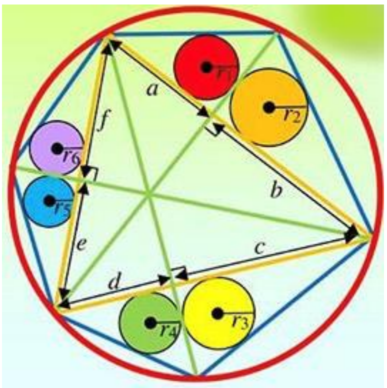
Yechimi: Quyidagi chizmaga qaraylik:



Bir xil yoyga tiralgan burchaklar tengligidan qizil rangli burchaklar tengligini topamiz. Demak sariq rangga bo‘yalgan to‘g‘ri burchakli uchburchaklar o‘zaro o‘xshash. U holda

$$\frac{r_B}{r_A}=\frac{l_2}{l_1} \quad \textbf{(19)}$$

nisbat bajariladi. Endi umumiyroq chizmaga qaraylik va belgilashlarni unga ko‘ra olaylik.



(19) munosabatga ko‘ra quyidagi tengliklarni topamiz:

$$\frac{r_1}{r_4}=\frac{a}{d},\ \frac{r_2}{r_5}=\frac{b}{e}\ \text{va}\ \frac{r_3}{r_6}=\frac{c}{f} \quad \textbf{(20)}$$

Shuningdek Cheva teoremasiga ko‘ra (yoki kesmalarning uzunliklarini ham qo‘yib topishimiz ham mumkin)

$$\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}=1 \quad \textbf{(21)}$$

U holda

$$\frac{r_1}{r_4}\cdot\frac{r_5}{r_2}\cdot\frac{r_3}{r_6}=\frac{a}{d}\cdot\frac{e}{b}\cdot\frac{c}{f}=1 \quad \textbf{(22)}$$

ya’ni

$$r_1r_3r_5=r_2r_4r_6 \quad \textbf{(23)}$$

tenglik isbotlandi.

7-9 sinf o‘quvchilari uchun

1. $x^3+x+1=2\sqrt{x^5+x+1}$ tenglamaning barcha musbat yechimlarini toping.

Yechimi: Tenglamaning har ikkala tarafini kvadratga oshirib, soddalashtirsak, berilgan tenglama quyidagi tenglamaga teng kuchli bo‘ladi:

$$x^2(x^4-4x^3+2x^2+4x+1)=0 \quad \textbf{(24)}$$

Bunga ko‘ra $x=0$ yechim ekanligi ravshan. Noldan farqli yechimlarni topish uchun

$$x^4-4x^3+2x^2+4x+1=0 \quad \textbf{(25)}$$

tenglamani yechish kerak. $x\neq 0$ ni hisobga olib har ikkala tomonini x^2 ga bo‘lsak,

$$x^2-4x+2+\frac{4}{x}+\frac{1}{x^2}=0 \quad \textbf{(26)}$$

tenglamani hosil qilamiz. Qulay usulda yechish uchun

$$x-\frac{1}{x}=m \quad \textbf{(27)}$$

belgilash kiritamiz. U holda

$$\left(x^2+\frac{1}{x^2}\right)-4\left(x-\frac{1}{x}\right)+2=0 \Rightarrow m^2+2-4m+2=0 \Rightarrow (m-2)^2=0 \quad \textbf{(28)}$$

ya'ni

$$x-\frac{1}{x}=2 \Rightarrow x_1=1+\sqrt{2} \text{ va } x_2=1-\sqrt{2} \quad \textbf{(29)}$$

yechimlarni topamiz. Masala to'liq yechildi.

Javob: $1-\sqrt{2}, 1+\sqrt{2}, 0$

2. $n^7+n^6+n^5+n^3+2n^2+2n+1$ ifoda tub son bo'ladigan barcha natural sonlarni toping.

Yechimi: Dastlab masala berilgan ifodani ko'paytuvchiga ajratamiz:

$$n^7+n^6+n^5+n^3+2n^2+2n+1=n^5(n^2+n+1)+n(n^2+n+1)+$$

$$(n^2+n+1)=(n^2+n+1)(n^5+n+1) \quad \textbf{(30)}$$

Bundan tashqari

$$n^2+n+1\geq 3 \quad \textbf{(31)}$$

va

$$n^5+n+1\geq 3 \quad \textbf{(32)}$$

tengsizliklarga ko'ra berilgan ifoda hech qachon tub son bo'lmaydi.

Javob: Bunday natural sonlar mavjud emas.

3. a, b turli haqiqiy sonlar berilgan. U holda $27ab(\sqrt[3]{a}+\sqrt[3]{b})^3=1$ bo'lishi uchun $27ab(a+b+1)=1$ bo'lishi zarur va yetarli ekanligini isbotlang.

Yechimi: Dastlab zaruriylik shartini isbotlaymiz.:

$$27ab(\sqrt[3]{a}+\sqrt[3]{b})^3=1 \Rightarrow (\sqrt[3]{a}+\sqrt[3]{b})^3=\frac{1}{27ab} \Rightarrow a+b+3\cdot\sqrt[3]{a}\cdot\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})=\frac{1}{27ab} \Rightarrow$$

$$\left\langle 3\cdot\sqrt[3]{a}\cdot\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})=\sqrt[3]{1}\right\rangle \Rightarrow a+b+1=\frac{1}{27ab} \Rightarrow 27ab(a+b+1)=1 \quad \textbf{(33)}$$

Endi yetarlilik shartini isbotlaymiz:

$$27ab(a+b+1)=1 \Rightarrow a+b+1=\frac{1}{27ab} \Rightarrow$$

$$\left(\sqrt[3]{a}+\sqrt[3]{b}\right)^3-3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})+1=\left(\frac{1}{3\sqrt[3]{a}\sqrt[3]{b}}\right)^3 \Rightarrow$$

$$\left(\sqrt[3]{a}+\sqrt[3]{b}\right)^3-\left(\frac{1}{3\sqrt[3]{a}\sqrt[3]{b}}\right)^3=3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})-1 \Rightarrow$$

$$\frac{\left(3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})\right)^3-1}{27ab}=3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})-1 \Rightarrow$$

$$\left(3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})-1\right)\left(\left(3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})\right)^2+3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})+1-27ab\right)=0 \quad \textbf{(34)}$$

Demak

$$3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})-1=0 \quad \textbf{(34)}$$

yoki

$$\left(3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})\right)^2+3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})+1-27ab=0 \quad \textbf{(35)}$$

Lekin

$$\left(3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})\right)^2 \ge \left(3\sqrt[3]{a}\sqrt[3]{b}\cdot 2\sqrt{\sqrt[3]{a}\sqrt[3]{b}}\right)^2=36ab>27ab$$

Tengsizlikka ko'ra **(35)** bajarilmaydi. Demak

$$27ab(a+b+1)=1 \Rightarrow 3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})=1 \Rightarrow 27ab(\sqrt[3]{a}+\sqrt[3]{b})^3=1.$$

4. Tashqi chizilgan aylanasining radiusi R ga teng bo'lgan o'tkir burchakli $\triangle ABC$ uchburchakning balandliklari H nuqtada kesishadi. Agar $AH=R$ tenglik bajarilsa, u holda $\angle BAC$ ni toping.

Yechimi: Aytaylik AD va BE kesmalar balandliklar bo'lsin. U holda

$$AH=\frac{AE}{\cos\angle CAD}=\frac{AE}{\cos(90^{\circ}-\angle C)}=\frac{AB\cos\angle A}{\sin\angle C}=\frac{2R\sin\angle C\cos\angle A}{\sin\angle C}=2R\cos\angle A \quad \textbf{(36)}$$

tenglikni topamiz. Masala shartiga ko‘ra

$$2R\cos\angle A=AH=R\Rightarrow\cos\angle A=\frac{1}{2}\Rightarrow\angle A=60^{\circ} \quad \textbf{(37)}$$

natijani topamiz.

Javob: $\angle A=60^{\circ}$.

5. $x^3+y^3+6xy=p+8$ tenglamaning barcha butun musbat yechimlarini toping, bu yerda p tub son.

Yechimi: Masala shartini quyidagicha yozib olamiz:

$$x^3+y^3+(-2)^3-3\cdot(-2)xy=p\Rightarrow(x+y-2)(x^2+y^2+4+2x+2y-xy)=p \quad \textbf{(38)}$$

U holda p tub son ekanligiga ko’ra

$$\begin{cases}x+y-2=1\\x^2+y^2+4+2x+2y-xy=p\end{cases}\Rightarrow\begin{cases}x+y=3\\3x^2-6x+19-p=0\end{cases}\Rightarrow$$

$$\begin{cases}x\in\{1,2\}\\3x^2-9x+19=p\end{cases}\Rightarrow\begin{cases}x=1\\p=13\end{cases}\cup\begin{cases}x=2\\p=13\end{cases}\Rightarrow(x,y,p)=(1,2,13);(2,1,13) \quad \textbf{(39)}$$

Javob: $(x,y,p)=(1,2,13);(2,1,13)$

4-6 sinf o‘quvchilari uchun

1. Gulruh 6 yoshda. Uning singlisi Mukambar undan 2 yosh kichkina, Uning akasi Nurmuhammad 2 yosh katta. Uchovining yoshlari yig‘indisini toping

A) 15 B) 16 C) 17 D) 18 E) 19

2. Quyonchada 20 ta sabzi bor. U har kuni 2 tadan sabzi yeydi. U 12-sabzini chorshanba kuni iste’mol qildi. U qaysi kuni sabzilarni yeyishni boshlagan?

A) Dushanda B) Seshanba C) Chorshanba D) Payshanba E) Juma

3. 13 ta bola musobaqaga registratsiyadan o‘tdi. So‘ngra yana 19 ta bola bu musobaqaga qo‘shildi. Musobaqada qatnashish uchun 6 ta teng sonli qatnashchilardan tuzilgan jamoalar kerak. Yana nechta bola musobaqaga qo‘shilishi kerak?

A) 1 B) 2 C) 3 D) 4 E) 5

4. Sarvar $5+4+3+2+1$ ifodani hisoblamoqda, Sardor esa $5-4-3-2-1$ ifodani son qiymatini topmoqda. Agar ularning javoblarini qo'shsak, necha chiqadi?

A) 10 B) 20 C) 15 D) 30 E) to'g'ri javob yo'q

5. Agar 16 ga 9 ni ko'paytirib, ko'paytmaga 15 va 14 sonlarining ko'paytmasini qo'shsak, hamda ushbu ifodani 6 ga bo'lsak, necha chiqadi?

A) 54 B) 57 C) 59 D) 56 E) to'g'ri javob yo'q

6. Imtihon 3 soat-u 40 daqiqa davom etdi. Agar imtihon 14:11 da boshlangan bo'lsa, soat nechada imtihon tugagan?

A) 17:41 B) 17:51 C) 18:51 D) 19:11 E) 10:51

7. Quyidagi ketma-ketlikdagi qonuniyatni aniqlab, keyingi hadini toping?

1, 1, 2, 6, 24, 120, …

A) 720 B) 840 C) 60 D) 148 E) 160

8. 6583 soniga qanday eng kichik musbat sonni qo'shsak, u 7 ga qoldiqsiz bo'linadi?

A) 2 B) 3 C) 4 D) 5 E) 6

9. 1 dan 88 gacha bo'lgan natural sonlar ketma-ket yozish natijasida yangi son hosil qilindi. Ushbu sonning raqamlari yig'indisini hisoblang.

A) 776 B) 768 C) 786 D) 748

10. Ikkita 6 tomonli o'yin toshini tashlaganda 36 ta vaziyat kuzatiladi, bu holatlarning nechtasida sonlar ko'paytmasi 7 ga qoldiqsiz bo'linadi?

A) 10 B) 8 C) 11 D) 9 E) to'g'ri javob yo'q

11. $124a85b$ soni 18 ga qoldiqsiz bo'linsa, u holda a va b raqamlar juftliklari sonini toping.

A) 5 B) 7 C) 6 D) 4

12. Abdullohga har haftada 5 so‘m beriladi, bunda u har kuni yarim so‘mni o‘zi uchun ishlatadi, qolganini yig‘adi. 6 haftadan keyin uning yig‘gan puli qanchaga yetadi?

A) 9 B) 12 C) 27 D) 6 E) to‘g‘ri javob yo‘q

13. Dastlabki 200 ta natural son ichida 5 ga ham, 3 ga ham bo‘linmaydiganlari nechta?

A) 85 B) 95 C) 75 D) 107

14. Hisoblang *5554·5558-5552·5556*

A) 20220 B) 22220 C) 20020 D) 22020

15. 8 ta disk uchun Abdurahmon 60 so‘m to‘ladi. Barcha disklar teng narxda edi, ammo u ikki diskni 25 foizga arzonga oldi. Standard disk necha so‘m turadi?

A) 4 B) 6 C) 8 D) 10 E) 9

Izoh: *Ba’zi testlarning berilishidagi xatoliklar tuzatildi.*

Test topshiriqlarining javoblari

1. D	**6. B**	**11. A**
2. E	**7. A**	**12. C**
3. D	**8. C**	**13. D**
4. A	**9. D**	**14. B**
5. C	**10. E**	**15. C**

Fan olimpiadalari bo‘yicha iqtidorli o‘quvchilar bilan ishlash departamenti sizga omadlar tilaydi!