Xalq Ta'limi vazirligi Fan olimpiadalari bo'yicha iqtidorli o'quvchilar bilan ishlash departamentining haftalik olimpiadasi topshiriqlari va yechimlari

10-11 sinf o'quvchilari uchun

1. Sistemani musbat haqiqiy sonlarda yeching:

$$\begin{cases} 27\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)}=8(x+y+z)^3 & (1) \\ x+y+z=\frac{1}{xyz} & (2) \end{cases}$$

Yechim: $yz=a,\ xz=b,\ xy=c$ kabi belgilaylik. U holda sistemaning 2-tenglamasiga ko'ra

$$ab+bc+ca=1 \qquad \textbf{(1.1)}$$

tenglik kelib chiqadi. 1-tenglamani quyidagchi soddalashtiramiz:

$$27\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)}=27\sqrt{\frac{(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)}{x^2y^2z^2}}=$$

$$27\sqrt{\frac{(a^2+1)(b^2+1)(c^2+1)}{abc}}=\langle ab+bc+ca=1 \text{ tenglikni inobatga olsak}\rangle=$$

$$27\sqrt{\frac{(a^2+ab+bc+ac)(b^2+ab+bc+ac)(c^2+ab+bc+ac)}{abc}}=$$

$$=27\sqrt{\frac{(a+b)(b+c)(b+a)(b+c)(c+a)(c+b)}{abc}}=27\sqrt{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}}=$$

$$27\sqrt{\frac{x^2y^2z^2(x+y)^2(y+z)^2(z+x)^2}{x^2y^2z^2}}=27(x+y)(y+z)(z+x) \qquad \textbf{(1.2)}$$

Demak sistemaning 1-tenglamasi quyidagicha ko'rinishga keladi:

$$27(x+y)(y+z)(z+x)=8(x+y+z)^3 \qquad \textbf{(1.3)}$$

Endi $y+z=A,\ z+x=B,\ x+y=C$ kabi belgilash olaylik. U holda **(1.3)** tenglama

$$(A+B+C)^3 = 27ABC \quad \textbf{(1.4)}$$

Ko'rinishiga keladi. Qavslarni ochib soddalashtirsak,

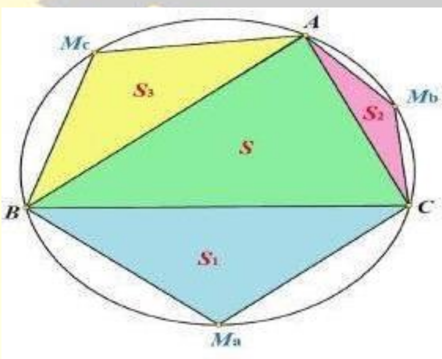
$$\frac{(A+B+C)((A-B)^2+(B-C)^2+(C-A)^2)}{2} + 3A(B-C)^2 + 3B(C-A)^2 + 3C(A-B)^2 = 0 \ \textbf{(1.4)}.$$

tenglikni topamiz.Demak $A=B=C$ yoki $x=y=z$ ekan. *(Siz agar o'rta qiymatlar haqidagi Koshi tengsizligini bilsangiz* ***(1.3)*** *tenglikni o'zidan $A=B=C$ tenglikni topishingiz mumkin).*

U holda (2) tenglamaga ko'ra $3x = \frac{1}{x^3}$ yoki $x=y=z=\frac{1}{\sqrt[4]{3}}$.

Javob: $x=y=z=\frac{1}{\sqrt[4]{3}}$

2. $\triangle ABC$ da R va r lar mos ravishda tashqi va ichki chizilgan aylanalar radiuslari, M_a, M_b, M_c lar yoy o'rtalari(rasmga qarang). U holda isbotlang:



$$\frac{S_1+S_2+S_3}{S} = \frac{R}{r} - 1$$

Yechim: Qulaylik uchun $BC=a,\ CA=b,\ AB=c$ kabi belgilaylik. Tashqi chizilgan aylana xossalariga ko'ra

$$BM_a = CM_a = 2R\sin\frac{\angle A}{2} \quad \textbf{(2.1)}$$

va

$$\angle BM_aC = 180^\circ - \angle A \quad \textbf{(2.2)}$$

tenglilarni hosil qilamiz. U holda

$$S_1 = \frac{1}{2}BM_a \cdot CM_a \cdot \sin(180^\circ - \angle A) = 2\sin^2\frac{\angle A}{2}R^2\sin\angle A =$$

$$R^2(1-\cos\angle A)\sin\angle A = R^2\cdot\frac{a}{2R} - \frac{R^2}{2}\cdot\sin 2\angle A = \frac{aR}{2} - \frac{R^2\sin 2\angle A}{2} \quad \textbf{(2.3)}$$

tenglikni topamiz. Xuddi shunday

$$S_2 = \frac{bR}{2} - \frac{R^2 \sin 2\angle B}{2} \quad \textbf{(2.4)}$$

va

$$S_3 = \frac{cR}{2} - \frac{R^2 \sin 2\angle C}{2} \quad \textbf{(2.5)}$$

tengliklarni ham topamiz. U holda

$$S_1 + S_2 + S_3 = \frac{(a+b+c)R}{2} - \frac{R^2(\sin 2\angle A + \sin 2\angle B + \sin 2\angle C)}{2} \quad \textbf{(2.6)}$$

Trigonametriyaning mashhur lemmalaridan biri

$$\sin 2\angle A + \sin 2\angle B + \sin 2\angle C = 4\sin\angle A \cdot \sin\angle B \cdot \sin\angle C \quad \textbf{(2.7)}$$

dan va uchburchak yuzasiga oid

$$S = \frac{a+b+c}{2} \cdot r = 2R^2 \sin\angle A \cdot \sin\angle B \cdot \sin\angle C \quad \textbf{(2.8)}$$

Formulalardan foydalanib,

$$\frac{S_1 + S_2 + S_3}{S} = \frac{R}{r} - \frac{2R^2 \sin\angle A \cdot \sin\angle B \cdot \sin\angle C}{S} = \frac{R}{r} - 1 \quad \textbf{(2.9)}$$

natijani topamiz. Masala to’liq yechildi.

3. $M = 12^{2^{15}} + 1$ sonining eng kichik tub bo’luvchisini toping.

Yechimi: Aytaylik $M = 12^{2^{15}} + 1$ ning eng kichik tub bo’luvchisi q bo’lsin. U holda

$$12^{2^{16}} - 1 = \left(12^{2^{15}} - 1\right)\left(12^{2^{15}} + 1\right) \vdots q \quad \textbf{(3.1)}$$

Demak

$$o_q(12) \mid 2^{16} \text{ yoki } o_q(12) = 2^{16} \quad \textbf{(3.2)}$$

Bundan tashqari Fermaning kichik teoremasiga ko’ra

$$12^{q-1} - 1 \vdots q \quad \textbf{(3.4)}$$

U holda

$$q - 1 \vdots o_q(12) = 2^{16} \quad \textbf{(3.5)}$$

Demak

$$2^{16} \le q-1 \text{ yoki } 2^{16}+1 \le q \quad \textbf{(3.6)}$$

Boshqa tomondan

$$2^{16}+1 = 2^{2^4}+1 = F_4 \quad \textbf{(3.7)}$$

Fermaning to’rtinchi tub soni.

Javob: $q = 2^{16}+1$.

4. Ko’paytmasi birga teng bo’lgan va hech biri birga teng bo’lmagan haqiqiy $x,\ y,\ z$ sonlar uchun quyidagi tengsizlikni isbotlang:

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2} \ge 1.$$

Yechimi: Quyidagicha belgilash olaylik:

$$\frac{x}{x-1}=a,\ \frac{y}{y-1}=b,\ \frac{z}{z-1}=c \quad \textbf{(4.1)}$$

U holda

$$\frac{a}{a-1}=x,\ \frac{b}{b-1}=y,\ \frac{c}{c-1}=z \quad \textbf{(4.2)}$$

bo’ladi. Masala shartiga ko’ra

$$xyz = \frac{abc}{(a-1)(b-1)(c-1)} = 1 \quad \textbf{(4.3)}$$

yoki

$$ab+bc+ac+1 = a+b+c \quad \textbf{(4.4)}$$

Masalada isbotlash talab qilingan tengsizlik

$$a^2+b^2+c^2 \ge 1 \quad \textbf{(4.5)}$$

tengsizlikka teng kuchli. **(4.4)** tenglikdan foydalansak oxirgi tengsizlik quyidagi tengsizlikka ekvivalent:

$$a^2+b^2+c^2+1-2(a+b+c-ab-bc-ca) \ge 0 \quad \textbf{(4.5)}$$

Bu tengsizlik esa

$$(a+b+c-1)^2 \ge 0 \quad \textbf{(4.6)}$$

tengsizlikka teng kuchli.

Izoh: *Ushbu masala 2008-yilgi Xalqaro Matematika Olimpiadasi(IMO) da foydalanilgan masalaning bir qismi.*

5. $\triangle ABC$ da AD va BE lar balandliklar, O tashqi chizilgan aylana markazi. Faraz qilaylik $O \in DE$ bo'lsin. U holda $\sin\angle A \cdot \sin\angle B \cdot \cos\angle C$ ning qiymatini toping.

Yechimi: Burchaklarni hisoblash orqali

$$\triangle CDE \text{ va } \triangle CAB$$

uchburchaklar o'xshashligini va

$$\angle OCD + \angle CDE = (90^\circ - \angle A) + \angle A = 90^\circ$$

tenglikni topamiz. U holda $O \in DE$ bo'lishi uchun $\triangle CDE$ ning balandligi CO ga teng bo'lishi kerak. Demak

$$R_{\triangle ABC} = CO = h(\triangle CDE; C) = h(\triangle CAB; C) \cdot \cos\angle C = AC \cdot \sin\angle A \cdot \cos\angle C =$$

$$2R_{\triangle ABC} \cdot \sin\angle B \cdot \sin\angle A \cdot \cos\angle C$$

yoki

$$\sin\angle A \cdot \sin\angle B \cdot \cos\angle C = \frac{1}{2}$$

tenglikni topamiz.

7-9 sinf o'quvchilari uchun

1. Agar $a^3 = a + 1$ bo`lsa, u holda $\sqrt[3]{3a^2 - 4a} + a\sqrt[4]{2a^2 + 3a + 2}$ ifodaning qiymatini toping.

Yechim: Quyidagi tengliklarni tekshirib ko'ramiz:

$$3a^2 - 4a = 3a^2 - 3a - (a^3 - 1) = (1 - a)^3 \qquad \textbf{(1.1)}$$

va

$$2a^6 + 3a^5 + 2a^4 = 2(a+1)^2 + 3a^2(a+1) + 2a^4 = 2a^4 + 3a^3 + 5a^2 + 4a + 2 =$$

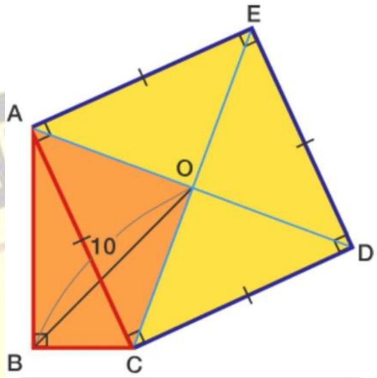
$$a^4 + (a+1)a + 3a^3 + 5a^2 + 3a + 1 + (a+1) = a^4 + 4a^3 + 6a^2 + 4a + 1 = (a+1)^4 \qquad \textbf{(1.2)}$$

Demak masalada so'ralgan ifoda quyidagiga teng bo'ladi:

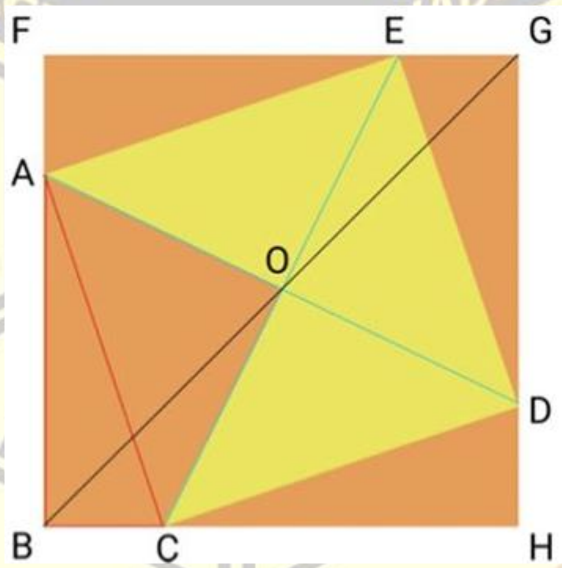
$$\sqrt[3]{3a^2-4a}+a\sqrt[4]{2a^2+3a+2}=\sqrt[3]{3a^2-4a}+\sqrt[4]{2a^6+3a^5+2a^4}=$$
$$\sqrt[3]{(1-a)^3}+\sqrt[4]{(1+a)^4}=1-a+1+a=2 \quad \textbf{(1.3)}$$

Javob: $\sqrt[3]{3a^2-4a}+a\sqrt[4]{2a^2+3a+2}=2$

2. Chizmadagi ma'lumotlarga asosan jigarrang sohaning yuzasini toping.



Yechim: Kvadrtaga quyidagi chizmadagi kabi bir uchi B nuqtada bo'lgan tashqi kvadrat chizaylik:



O'zaro teng to'g'ri burchakli uchburchaklarga ko'ra tashqi chizilgan katta kvadrat tomoni uzunligi $BC+AB=d$ ga teng. Masala shartiga ko'ra

$$BG=2\cdot BO=20 \quad \textbf{(2.1)}$$

Demak

$$2d^2=FB^2+FG^2=BG^2=400 \quad \textbf{(2.2)}$$

Ya'ni

$$d=\sqrt{200}=10\sqrt{2} \quad \textbf{(2.3)}$$

U holda katta kvadratning yuzasi

$$S_{FBHG} = d^2 = 200 \qquad \textbf{(2.4)}$$

Boshqa tomondan katta kvadrat $OABC$ to’rtburchak bilan tengdosh bo’lgan to’rtta to’rtburchakdan iborat, ya’ni

$$S_{OABC} = \frac{200}{4} = 50 \qquad \textbf{(2.5)}$$

Javob: $S_{OABC} = 50$.

3. Eng kichik 4 xonli sonni topingki, $\overline{abcd} = \overline{ab} \cdot \overline{cd} + \overline{ab} + \overline{cd}$ tenglik o’rinli bo`lsin.

Yechimi: $x = \overline{ab}$ va $y = \overline{cd}$ kabi belgilaylik. U holda

$$100x + y = \overline{abcd} = xy + x + y \qquad \textbf{(3.1)}$$

Demak

$$100 = y + 1 \text{ yoki } \overline{cd} = y = 99 \qquad \textbf{(3.2)}$$

Ko’rishimiz mumkinki $x = \overline{ab}$ ixtiyoriy ikki xonali son bo’lishi mumkin. Demak $\overline{abcd}$ eng kichik son bo’lishi uchun

$$x = \overline{ab} = 10 \qquad \textbf{(3.4)}$$

bo’lishi zarur.U holda

Javob: $\overline{abcd}_{\min} = 1099$.

4. Kamida *2* ta raqami bir xil o`n xonali sonlar nechta?

Yechimi: Masalani “teskari tomondan kelish” usuli orqali yechamiz. Ravshanki barcha o’n xonali sonlar $9 \cdot 10^9$ ta. Endi barcha raqami turlicha bo’lgan o’n xonali sonlar sonini topaylik:

$$\begin{array}{cccccccccc} \multicolumn{10}{c}{\overline{a_{10}a_9a_8a_7a_6a_5a_4a_3a_2a_1}} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 9 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \qquad \textbf{(4.1)}$$

Chunki

$$a_{10} \neq 0 \;\; a_9 \neq a_{10} \;\; a_8 \neq a_9, a_{10} \ldots a_1 \neq a_2, a_3, \ldots a_{10}$$

Ya’ni ularning soni

$$9 \cdot 9! \qquad \textbf{(4.2)}$$

ta ekan. U holda kamida 2 ta raqami bir xil bo’lgan o’n xonali sonlar soni

$$9(10^9 - 9!) \quad \textbf{(4.3)}$$

Javob: $9(10^9 - 9!)$ ta.

5. $\sqrt[5]{2} + 7$ va $8\sqrt[10]{2}$ ni taqqoslang.

Yechimi: $x = \sqrt[10]{2}$ deb belgilaylik. U holda biz quyidagi sonlarni taqqoslashimiz kerak.

$$x^2 + 7 \text{ va } 8x \quad \textbf{(5.1)}$$

Ularning ayirmasini qaraylik:

$$x^2 + 7 - 8x = (x-1)(x-7) \quad \textbf{(5.2)}$$

Ma'lumki,

$$\begin{cases} x = \sqrt[10]{2} > 1 \\ x = \sqrt[10]{2} < 7 \end{cases} \quad \textbf{(5.3)}$$

Ya'ni

$$x^2 + 7 - 8x = (x-1)(x-7) < 0 \quad \textbf{(5.4)}$$

Demak

Javob: $\sqrt[5]{2} + 7 < 8\sqrt[10]{2}$.

4-6 sinf o'quvchilari uchun

1. BIMC+BI+MC+M+I+M+1=2020, bunda B, I, M, C raqamlar bo'lsa, 4 xonali BIMC sonining eng katta qiymatini toping.

Yechim: Tenglikning chap tomonini xonalarga ajratib yozib olamiz:

BIMC+BI+MC+M+I+M+1=1000B+100I+10(2M+B)+2(M+C+I)+1=2020 **(1.1)**

Ko'rishimiz mumkinki tenglamaning chap tomoni toq son va o'ng tomoni juft son. Demak bunday to'rt xonali son mavjud emas.

2. Uchta 3 raqami va arifmetik amallardan foydalanib 1, 2, 3, 4 sonlarini hosil qiling. *(Masalan 3:3+3=4)*

Yechim: Quyidagi amallar orqali masala o'z yechimini topadi:

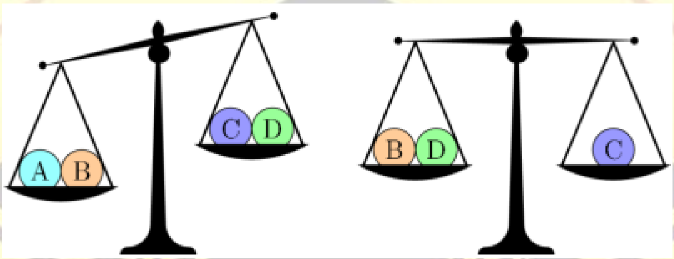
$$1 = 3!:(3+3) = 3^{3-3}$$
$$2 = (3+3):3 = 3-3:3$$
$$3 = 3+3-3 = 3\cdot 3:3$$
$$4 = 3:3+3$$

3. 9 ta ot 3 kunda 27 bog' o't yeydi. U holda 5 ta ot 5 kunda nechta bog' o't yeydi.

Yechimi: Masala shartidan ayonki, 1 ta ot 1 kunda $(27:3):9=1$ ta bog' o't yeydi. Demak 5 ta ot 5 kunda $(1\cdot 5)\cdot 5=25$ bog' o't yerkan.

Javob:25 bog'.

4. 4 ta koptok mos ravishda 10, 20, 30 va 40 g og'irlikka ega. Qaysi koptok 30 g?



Yechimi: Masala shartiga ko'ra

$$C+D<A+B \quad \textbf{(4.1)}$$

va

$$B+D=C \quad \textbf{(4.2)}$$

Munosabatlar o'rinli. Bu ikki munosabatni qo'shib,

$$2D<A \quad \textbf{(4.2)}$$

Tengsizlikni topamiz. Demak

$$B+10=C \text{ va } 20<A \quad \textbf{(4.2)}$$

U holda biz quyidagi ikki holni qarashimiz yetarli:

1-hol: $B=20$ bo'lsin. Bu holda

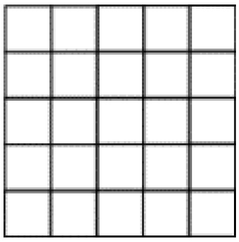
$$C=30 \Rightarrow A=40$$

2-hol: $B=30$ bo'lsin. Bu holda

$$C=40 \Rightarrow A=10 \Rightarrow \varnothing$$

Javob: $C=30$.

5. Quyidagi shaklda nechta kvadrat bor?



Yechimi: Tomonlari uzunliklari bo'yicha guruhlab kvadratlar sonini topamiz:

Tomoni uzunligi 1 ga teng bo'lgan kvadratlar soni: $5 \cdot 5$ ta

Tomoni uzunligi 2 ga teng bo'lgan kvadratlar soni: $4 \cdot 4$ ta

Tomoni uzunligi 3 ga teng bo'lgan kvadratlar soni: $3 \cdot 3$ ta

Tomoni uzunligi 4 ga teng bo'lgan kvadratlar soni: $2 \cdot 2$ ta

Tomoni uzunligi 5 ga teng bo'lgan kvadratlar soni: $1 \cdot 1$ ta

Demak shakldagi barcha kvadratlar soni

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55 \text{ ta} \quad \textbf{(5.1)}$$

Javob: 55 ta.

Izoh: *Agar shakl tomoni n ga teng bo'lgan kvadrat bo'lsa, u holda shakldagi kvadratlar soni*

$$\frac{n(n+1)(2n+1)}{6}$$

ta chiqadi.